

Dynamical critical phenomena in driven-dissipative systems

L. M. Sieberer^{1,2}, S. D. Huber^{3,4}, E. Altman^{4,5}, and S. Diehl^{1,2}

¹*Institute for Theoretical Physics, University of Innsbruck, A-6020 Innsbruck, Austria*

²*Institute for Quantum Optics and Quantum Information of the Austrian Academy of Sciences, A-6020 Innsbruck, Austria*

³*Theoretische Physik, Wolfgang-Pauli-Strasse 27, ETH Zurich, CH-8093 Zurich, Switzerland*

⁴*Department of Condensed Matter Physics, Weizmann Institute of Science, Rehovot 76100, Israel*

⁵*Department of Physics, University of California, Berkeley, CA 94720, USA*

We explore the nature of the Bose condensation transition in driven open quantum systems, such as exciton-polariton condensates. Using a functional renormalization group approach formulated in the Keldysh framework, we characterize the dynamical critical behavior that governs decoherence and an effective thermalization of the low frequency dynamics. We identify a critical exponent special to the driven system, showing that it defines a new dynamical universality class. Hence critical points in driven systems lie beyond the standard dynamical classification of equilibrium phase transitions. We show how the new critical exponent can be probed in experiments with driven cold atomic systems and exciton-polariton condensates.

Recent years have seen major advances in the exploration of many-body systems in which matter is strongly coupled to light [1]. Such systems include for example polariton condensates [2], superconducting circuits coupled to microwave resonators [3, 4], cavity quantum electrodynamics [5] as well as ultracold atoms coupled to high finesse optical cavities [6]. As in traditional quantum optics settings, these experiments are subject to losses, which may be compensated by continuous drive, yet they retain the many-body character of condensed matter. This combination of ingredients from atomic physics and quantum optics in a many-body context defines a qualitatively new class of quantum matter far from thermal equilibrium. An intriguing question from the theoretical perspective is what new universal behavior can emerge under such conditions.

A case in point are exciton-polariton condensates. Polaritons are short lived optical excitations in semiconductor quantum wells. Continuous pumping is required to maintain their population in steady state. But in spite of the non-equilibrium conditions, experiments have demonstrated Bose condensation [2] and, more recently, have even observed the establishment of a critical phase with power-law correlations in a two dimensional system below a presumed Kosterlitz-Thouless phase transition [7]. At a fundamental level however there is no understanding of the condensation transition in the presence of loss and external drive, and more generally of continuous phase transitions under such conditions.

In this letter we develop a theory of dynamical critical phenomena in driven-dissipative systems in three dimensions. Motivated by the experiments described above we focus on the case of Bose condensation. It is a common belief that such systems develop an effective temperature in steady state, and that phase transitions in driven-dissipative systems mimic in all aspects classical dynamical critical behavior at finite temperature. In certain cases it is observed that an effective temperature in-

deed develops [8–11]. We show below that while static properties of the driven-critical condensate are indeed governed by an effective temperature, the dynamical behavior belongs to a different universality class. To this end, we relate and highlight the differences between the dynamics of a driven system near a critical point, and the well-known classification of dynamical critical phenomena around equilibrium phase transitions [12]. In particular, we identify a measurable universal exponent that betrays the driven nature of the system and would not be obtained in the dynamic response at equilibrium critical points.

Open system dynamics– A microscopic description of driven open systems typically starts from a Markovian quantum master equation or an equivalent Keldysh action. However, the novel aspects in the critical dynamics of driven dissipative systems discussed below can be most simply illustrated by considering an effective phenomenological description of the order parameter dynamics using a stochastic Gross-Pitaevskii equation

$$i\partial_t\psi = \left[-(A - iD)\nabla^2 - \mu - i\chi + (\lambda - i\kappa)|\psi|^2 \right] \psi + \zeta. \quad (1)$$

As we show below, this equation can be rigorously derived from a fully quantum microscopic description of the condensate when including only the relevant terms near the critical point. The different terms in (1) have a clear physical origin. $\chi = \gamma_p - \gamma_l$ is the effective gain, which combines the incoherent pump field minus the local single-particle loss terms. $\kappa, \lambda > 0$ are respectively two-body loss and interaction parameters. The diffusion term D is not contained in the original microscopic model, but is generated upon integrating out high frequency modes. We therefore include it at the outset with a phenomenological coefficient. Finally ζ is a Gaussian white noise with correlations $\langle \zeta^*(t, \mathbf{x}) \zeta(t', \mathbf{x}') \rangle = \gamma \delta(t - t') \delta(\mathbf{x} - \mathbf{x}')$. Such noise is necessarily induced by the losses, and sudden appearances of particles due to pumping.

Without the noise term, Eq. (1) is similar to the dissipative Gross-Pitaevskii (dGP) equation previously used as a mean field description of the exciton-polariton condensates [13, 14]. The dGP admits a stationary condensate solution with density $|\psi|^2 = -\chi/\kappa$ for $\chi < 0$, undergoing a mean field transition to the vacuum state when χ crosses zero. The chemical potential of the condensate $\mu = \lambda|\psi|^2$ is determined by dynamical stability [15]. We also note that the dGP equation used in [13, 14] does not feature the diffusion term D mentioned above, which is expected to play a role near the critical point.

The noise term in Eq. (1) acts similarly to a temperature, in that it can drive a transition at finite particle density, thereby inducing critical fluctuations. As the equation of motion is cast in Langevin form, one might suspect that it can be categorized into one of the well-known models of dynamical critical phenomena classified by Hohenberg and Halperin [12]. However, this is not true in general. The models in Ref. [12] are constrained to have a very specific relation between the reversible and dissipative terms to ensure a thermal Gibbs distribution of the fields in steady state [16, 17]. This is equivalent to the requirement that both the coherent and dissipative evolutions are generated by the same Hamiltonian. In the driven open systems we consider, there is no such requirement. The dissipative dynamics is determined by the intensity of the pump and loss terms, independently of the intrinsic Hamiltonian dynamics of the system. For the critical dynamics this implies that the scaling of the dissipative coupling constants is not necessarily tied to that of the reversible ones by the equilibrium constraint, opening up the possibility for novel dynamical universality classes.

Microscopic Model – Having illustrated the nature of the problem with the effective classical equation (1) we turn to a fully quantum description within the Keldysh framework. Our starting point is the Markovian dissipative Keldysh action [18]

$$\mathcal{S} = \int_{t,x} \left\{ (\phi_c^*, \phi_q^*) \begin{pmatrix} 0 & P^A \\ P^R & P^K \end{pmatrix} \begin{pmatrix} \phi_c \\ \phi_q \end{pmatrix} + 2i\kappa\phi_c^*\phi_c\phi_q^*\phi_q \right. \\ \left. - \frac{1}{2} [(\lambda + i\kappa)(\phi_c^{*2}\phi_c\phi_q + \phi_q^{*2}\phi_c\phi_q) + c.c.] \right\}. \quad (2)$$

Here ϕ_c, ϕ_q are the so-called classical and quantum fields, given by the symmetric and anti-symmetric combinations of the fields on the forward and backward parts of the Keldysh contour [19, 20]. The inverse Green's functions are given by $P^R = i\partial_t + A\nabla^2 + \mu + i\chi$, $P^A = P^{R\dagger}$, $P^K = i\gamma$.

Vanishing of the mass scale χ defines a Gaussian fixed point with dynamical critical exponent $z = 2$ ($\omega \sim k^z$). Canonical power counting determines the scaling dimensions of the fields and interaction constants with respect to this fixed point: at criticality, the spectral components of the Gaussian action scale as $P^{R/A} \sim k^2$, while the

Keldysh component generically takes a constant value, i.e., $P^K \sim k^0$. Hence, to maintain scale invariance of the quadratic action, the scaling dimensions of the fields must be $[\phi_c] = \frac{d-2}{2}$ and $[\phi_q] = \frac{d+2}{2}$. From this result we read off the canonical scaling dimensions of the interaction constants; in particular, for a local interaction vertex containing n_c classical and n_q quantum fields, we find $[\lambda_{n_c, n_q}] = d + 2 - n_c \frac{d-2}{2} - n_q \frac{d+2}{2}$. Hence, the upper critical dimension for the quartic couplings is $d = 4$ as in finite temperature thermodynamic equilibrium. Moreover, in the case of interest $d = 3$, local vertices containing more than two quantum fields ($n_q \geq 1$ is required by causality [19, 20]) or more than five classical fields are irrelevant. The only marginal term with $n_q = 2$ is the Keldysh component of the single-particle inverse Green's function, i.e., the noise vertex. In this sense, the critical theory is equivalent to a stochastic *classical* problem [21, 22], as previously observed in [8, 23]. But as noted above it cannot be *a priori* categorized in one of the dynamical universality classes [12] subject to an intrinsic equilibrium constraint.

Functional RG – To focus on the critical behavior we turn to a functional RG formulation for the field theory (2). The original equilibrium formulation due to Wetterich [24] was adapted to the Keldysh real time framework in [25, 26]. Central to this approach is the functional $\Gamma_\Lambda[\phi_c, \phi_q]$ defined by [27]

$$e^{i\Gamma_\Lambda[\phi_c, \phi_q]} = \int \mathcal{D}\phi_c \mathcal{D}\phi_q e^{i\mathcal{S}[\phi_c + \delta\phi_c, \phi_q + \delta\phi_q] + i\Delta\mathcal{S}_\Lambda[\delta\phi_c, \delta\phi_q]}. \quad (3)$$

Here $\Delta\mathcal{S}_\Lambda$ is a regulator function which suppresses contributions to the above path integral from modes with spatial wave-vector below the running cutoff Λ (see SI for details). Thus Γ_Λ interpolates between the classical action \mathcal{S} , when Λ equals the UV cutoff Λ_0 , and the effective action functional $\Gamma[\phi_c, \phi_q]$ [28] when $\Lambda \rightarrow 0$. The latter includes the effects of fluctuations on all scales. The equation

$$\partial_\Lambda \Gamma_\Lambda = \frac{i}{2} \text{Tr} \left[\left(\Gamma_\Lambda^{(2)} + R_\Lambda \right)^{-1} \partial_\Lambda R_\Lambda \right] \quad (4)$$

describes the flow of the interpolating functional as a function of the running cutoff Λ . Here $\Gamma_\Lambda^{(2)}$ and R_Λ denote the second field variations of Γ_Λ and $\Delta\mathcal{S}_\Lambda$ respectively, which makes Eq. (4) a functional differential equation. For the description of general equilibrium [29] and Ising dynamical [30] critical behavior the functional RG gave results that are competitive with high-order epsilon expansion and with Monte Carlo simulations already in rather simple approximation schemes.

We solve Eq. (4) approximately using an ansatz for Γ_Λ which includes all couplings that are relevant or marginal at tree level. Specifically, we take

$$\Gamma_\Lambda = \int_{t,x} \left\{ (\phi_c^*, \phi_q^*) \left(\begin{array}{c} 0 \\ iZ^* \partial_t + \bar{K}^* \nabla^2 \\ i\bar{\gamma} \end{array} \right) \begin{pmatrix} \phi_c \\ \phi_q \end{pmatrix} - \left(\frac{\partial \bar{U}}{\partial \phi_c} \phi_q + \frac{\partial \bar{U}^*}{\partial \phi_c^*} \phi_q^* \right) \right\}. \quad (5)$$

The fact that the spectral components of the effective action depend only linearly on ϕ_q allowed us to introduce an effective potential \bar{U} determined by the static couplings. $\bar{U}(\rho_c) = \frac{1}{2} \bar{u} (\rho_c - \rho_0)^2 + \frac{1}{6} \bar{u}' (\rho_c - \rho_0)^3$ is a function of the $U(1)$ invariant combination of classical fields $\rho_c = \phi_c^* \phi_c$ alone. It has a mexican hat structure ensuring dynamical stability. With this choice we approach the transition from the ordered side, taking the limit of the stationary state condensate $\rho_0 = \phi_c^* \phi_c|_{\text{st}} = \phi_0^* \phi_0 \rightarrow 0$.

All the parameters appearing in (5) including the stationary condensate density ρ_0 are functions of the running cutoff Λ . Hence, the functional flow of Γ_Λ is reduced by means of the approximate ansatz to the flow of a finite number of coupling constants $\mathbf{g} = (Z, \bar{K}, \rho_0, \bar{u}, \bar{u}', \bar{\gamma})^T$ determined by the β -functions $\Lambda \partial_\Lambda \mathbf{g} = \beta_{\mathbf{g}}(\mathbf{g})$ (see SI). The critical system is described by a scaling solution to these flow equations. In practice, it is obtained as a fixed point of the flow of dimensionless (according to the above power counting) renormalized couplings, which we derive in the following. First we rescale couplings with Z ,

$$K = \bar{K}/Z, \quad u = \bar{u}/Z, \quad u' = \bar{u}'/Z, \quad \gamma = \bar{\gamma}/|Z|^2. \quad (6)$$

Coherent and dissipative processes are encoded, respectively, in the real and imaginary parts of the renormalized kinetic coefficient $K = A + iD$ and the two- and three-body couplings $u = \lambda + i\kappa$ and $u' = \lambda' + i\kappa'$. The relative strength of these contributions is thus measured by the (dimensionless) ratios $r_K = A/D$, $r_u = \lambda/\kappa$, and $r_{u'} = \lambda'/\kappa'$, and we will find it convenient to work with these ratios instead of the respective real parts. The two- and three-body loss coefficients κ and κ' and the condensate density ρ_0 become dimensionless by means of the rescalings

$$w = \frac{2\kappa\rho_0}{\Lambda^2 D}, \quad \tilde{\kappa} = \frac{\gamma\kappa}{2\Lambda D^2}, \quad \tilde{\kappa}' = \frac{\gamma^2\kappa'}{4D^3}. \quad (7)$$

The flow equations for the couplings $\mathbf{r} = (r_K, r_u, r_{u'})^T$ and $\mathbf{s} = (w, \tilde{\kappa}, \tilde{\kappa}')^T$ form a closed set,

$$\Lambda \partial_\Lambda \mathbf{r} = \beta_{\mathbf{r}}(\mathbf{r}, \mathbf{s}), \quad \Lambda \partial_\Lambda \mathbf{s} = \beta_{\mathbf{s}}(\mathbf{r}, \mathbf{s}) \quad (8)$$

(see SI for the explicit form). As a consequence of the transformations (6) and (7), these β -functions acquire a contribution from the running anomalous dimensions $\eta_a(\mathbf{r}, \mathbf{s}) = -\Lambda \partial_\Lambda \ln a$ associated with $a = Z, D, \gamma$.

Critical properties – The universal behavior near the critical point is controlled by the infrared flow to a Wilson-Fisher like fixed point. The values of the coupling constants at the fixed point, determined by solving $\beta_{\mathbf{r}}(\mathbf{r}_*, \mathbf{s}_*) = 0$ and $\beta_{\mathbf{s}}(\mathbf{r}_*, \mathbf{s}_*) = 0$, are given by:

$$\begin{aligned} \mathbf{r}_* &= (r_{K*}, r_{u*}, r_{u'*}) = \mathbf{0}, \\ \mathbf{s}_* &= (w_*, \tilde{\kappa}_*, \tilde{\kappa}'_*) \approx (0.475, 5.308, 51.383). \end{aligned} \quad (9)$$

The first line implies that the fixed point action is purely imaginary (or dissipative), as in Model A of Hohenberg and Halperin [12], cf. Fig. 1 (c). We interpret the fact that the ratios of coherent vs. dissipative couplings are zero at the fixed point as a manifestation of decoherence at low frequencies in an RG framework. The coupling values \mathbf{s}_* are identical to those obtained in an equilibrium classical $O(2)$ model from functional RG calculations at the same level of truncation [29].

Let us turn to the linearized flow, which determines the universal behavior in the vicinity of the fixed point. We find that the two sectors corresponding to \mathbf{s} and \mathbf{r} decouple in this regime, giving rise to a block diagonal stability matrix

$$\frac{\partial}{\partial \ln \Lambda} \begin{pmatrix} \delta \mathbf{r} \\ \delta \mathbf{s} \end{pmatrix} = \begin{pmatrix} N & 0 \\ 0 & S \end{pmatrix} \begin{pmatrix} \delta \mathbf{r} \\ \delta \mathbf{s} \end{pmatrix}, \quad (10)$$

where $\delta \mathbf{r} \equiv \mathbf{r}$, $\delta \mathbf{s} \equiv \mathbf{s} - \mathbf{s}_*$, and N, S are 3×3 matrices (see SI).

The anomalous dimensions entering this flow are found by plugging the fixed point values $\mathbf{r}_*, \mathbf{s}_*$ into the expressions for $\eta_a(\mathbf{r}, \mathbf{s})$. We obtain the scaling relation between the anomalous dimensions $\eta_Z = -\eta_\gamma$, valid in the universal infrared regime. This leads to cancellation of η_Z with η_γ in the static sector S (see SI). The critical properties in this sector, encoded in the eigenvalues of S , become identical to those of the standard $O(2)$ transition. This includes the correlation length exponent $\nu \approx 0.716$ and the anomalous dimension $\eta \approx 0.039$ associated with

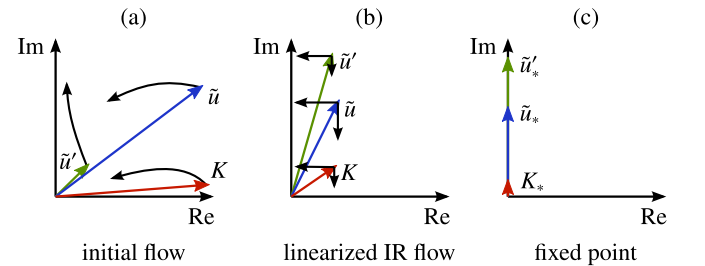


FIG. 1. Flow in the complex plane of dimensionless renormalized couplings. (a) The microscopic action determines the initial values of the flow. Typically, the coherent propagation will dominate over the diffusion, $A \gg D$, while two-body collisions and two-body loss are on the same order of magnitude, $\tilde{\lambda} \approx \tilde{\kappa}$, with a similar relation for the marginal complex coupling \tilde{u}' . The initial flow is non-universal. (b) At criticality, the infrared (IR) flow approaches a universal (independent of the initial conditions) linear domain encoding the critical exponents and anomalous dimensions. (c) The Wilson-Fisher fixed point describing the interacting critical system is purely imaginary.

the bare kinetic coefficient \bar{K} . These values are in good agreement with more sophisticated approximations [31].

The equilibrium-like behavior in the S sector can be seen as a result of an emergent symmetry. Locking of the noise to the dynamical term implied by the scaling relation $\eta_Z = -\eta_\gamma$ leads to invariance of the long wavelength effective action (times i) under the transformation $\Phi_c(t, \mathbf{x}) \rightarrow \Phi_c(-t, \mathbf{x})$, $\Phi_q(t, \mathbf{x}) \rightarrow \Phi_q(-t, \mathbf{x}) + \frac{2}{\gamma} \sigma^z \partial_t \Phi_c(-t, \mathbf{x})$, $i \rightarrow -i$ with $\Phi_\nu = (\phi_\nu, \phi_\nu^*)^T$, $\nu = (c, q)$, σ^z the Pauli matrix. This symmetry is a direct manifestation of a classical fluctuation-dissipation relation. It generalizes the symmetry noted in Refs. [32, 33] to models that include also reversible couplings compatible with an equilibrium stationary state. Unlike such Hohenberg-Halperin type models here the symmetry is not imposed at some mesoscopic level of the theory, but rather is emergent at the critical point. We interpret this finding as an effective low-frequency thermalization mechanism of the driven system at criticality.

The key new element in the driven-dissipative dynamics is encoded in the decoupled “drive” sector (the 3×3 matrix N in our case). It describes the flow towards the emergent purely dissipative Model A fixed point (see Fig. 1 (b)) and thus reflects a mechanism of low frequency decoherence. This sector has no counterpart in the standard framework of dynamical critical phenomena and is special to driven-dissipative systems. In the deep infrared regime, only the lowest eigenvalue of this matrix governs the flow of the ratios. This means that only one new critical exponent $\eta_r \approx -0.101$ is encoded in this sector; below we indicate how it affects single particle observables. Just as the dynamical critical exponent z is independent of the static ones, the block diagonal structure of the stability matrix ensures that the drive exponent is independent of the exponents of the other sectors.

The fact that the inverse Green’s function in Eq. (5) is specified by three real parameters, $\text{Re } \bar{K}$, $\text{Im } \bar{K}$, and $|Z|$ (the phase of Z can be absorbed by a $U(1)$ transformation) allows for only the three independent anomalous dimensions: η_D , η_Z and the new exponent η_r . Hence the extension of critical dynamics described here is *maximal*, i. e., no further independent exponent will be found. Moreover this extension of the purely relaxational (Model A) dynamics leads to different universality than an extension that adds reversible couplings compatible with relaxation towards a Gibbs ensemble. The latter is obtained by adding real couplings to the imaginary ones with the same ratio of real to imaginary parts for all couplings [34–37]. Such an extension adds only an independent 1×1 sector N to the purely relaxational problem, for which we find $\eta_R = -0.143 \neq \eta_r$. This proves that the independence of dissipative and coherent dynamics defines indeed a new non-equilibrium universality class with no equilibrium counterpart.

Experimental detection – The novel anomalous dimension identified here leaves a clear fingerprint in single-

particle observables accessible with current experimental technologies on different platforms. Using the RG scaling behavior of the diffusion and propagation coefficients $D \sim \Lambda^{-\eta_D}$, $A = Dr_K \sim \Lambda^{-\eta_r - \eta_D}$, we obtain the anomalous scaling of the frequency and momentum resolved, renormalized retarded Green’s function $G^R(\omega, \mathbf{q}) = (\omega - A_0 |\mathbf{q}|^{2-\eta_r - \eta_D} + iD_0 |\mathbf{q}|^{2-\eta_D})^{-1}$, with A_0 and D_0 non-universal constants. The peak position and width implied by the complex dispersion $\omega \approx A_0 |\mathbf{q}|^{2.22} - iD_0 |\mathbf{q}|^{2.12}$ can be measured in ultra-cold atomic systems by RF spectroscopy [38]. In exciton-polariton condensates, the dispersion relation can be reconstructed using homodyne detection techniques [39], which allows one to resolve both real and imaginary part of the retarded response separately.

Finally, we estimate the extent of the universal critical domain governed by the linearized regime of the Wilson-Fisher fixed point, which provides us with an estimate of the energy resolution χ_c necessary to probe the critical behavior. This is done by calculating the Ginzburg scale, i. e., the distance from the phase transition where fluctuations become dominant [28]: We equate the bare distance from the phase transition χ to the corresponding one-loop correction, yielding $\chi_c = 8\pi\gamma\kappa/D^{3/2}$. The diffusion constant D can be obtained from experiment by the same techniques as already indicated. A two-loop estimate gives the scaling $D \sim \lambda^2 n^2, \kappa^2 n^2$.

Conclusions – We have developed a Keldysh field theoretical approach to characterize the critical behavior of driven-dissipative three dimensional Bose condensates at the condensation transition. The main result presents a hierarchical extension of classical critical phenomena. First, all static aspects are identical to the classical $O(2)$ critical point. In the next shell of the hierarchy a sub-class of the dynamical phenomena is identical to the purely dissipative model A dynamics of the equilibrium critical point. Finally we identify manifestly non-equilibrium features of the critical dynamics, encoded in a new independent critical exponent that betrays the driven nature of the system.

It is an interesting question for future work how the above conclusions are modified when applied to systems of lower dimensionality. Particularly pertinent for understanding current experiments with exciton-polariton condensates in semiconductor quantum wells is the case of two dimensional systems.

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Supplementary Information for “Dynamical critical phenomena in driven-dissipative systems”

L. M. Sieberer^{1,2}, S. D. Huber^{3,4}, E. Altman^{4,5}, and S. Diehl^{1,2}

¹*Institute for Theoretical Physics, University of Innsbruck, A-6020 Innsbruck, Austria*

²*Institute for Quantum Optics and Quantum Information of the Austrian Academy of Sciences, A-6020 Innsbruck, Austria*

³*Theoretische Physik, Wolfgang-Pauli-Strasse 27, ETH Zurich, CH-8093 Zurich, Switzerland*

⁴*Department of Condensed Matter Physics, Weizmann Institute of Science, Rehovot 76100, Israel*

⁵*Department of Physics, University of California, Berkeley, CA 94720, USA*

FUNCTIONAL RENORMALIZATION GROUP EQUATION

Our approach to studying critical dynamics is based on the Wetterich functional renormalization group equation [1] adapted to the Keldysh framework

$$\partial_\Lambda \Gamma_\Lambda = \frac{i}{2} \text{Tr} \left[\left(\Gamma_\Lambda^{(2)} + R_\Lambda \right)^{-1} \partial_\Lambda R_\Lambda \right]. \quad (\text{SI-1})$$

In the first part of this supplement we discuss the objects appearing in (SI-1), namely the second functional derivative $\Gamma_\Lambda^{(2)}$ and the cutoff function R_Λ . Next we explain how a closed set of flow equations for a finite number of coupling constants can be obtained from the functional flow equation. Finally we detail the linearized equations for the infrared flow to the Wilson-Fisher fixed point from which the critical properties are inferred.

In suitable truncation schemes, results from high order epsilon expansion can be reproduced from the exact flow equation (SI-1). In our practical calculation, we approach the critical point from the ordered phase. This allows us to calculate the anomalous dimensions at one-loop order, due to the presence of a finite condensate during the flow. Results obtained in this way have proven to be competitive with high-order epsilon expansion or Monte Carlo simulations, as referenced in the main text.

THE SECOND VARIATIONAL DERIVATIVE

The second variation $\Gamma_\Lambda^{(2)}$ with respect to the fields is the full inverse Green’s function at the scale Λ , which in the case of an interacting theory is field dependent. Practically we work in a basis of real fields, related to the complex fields by

$$\begin{pmatrix} \chi_{\nu,1}(Q) \\ \chi_{\nu,2}(Q) \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix} \begin{pmatrix} \phi_\nu(Q) \\ \phi_\nu^*(-Q) \end{pmatrix}, \quad (\text{SI-2})$$

where $\nu = c, q$ is the Keldysh index. We gather the resulting four independent field components in a field vector,

$$\chi(Q) = (\chi_{c,1}(Q), \chi_{c,2}(Q), \chi_{q,1}(Q), \chi_{q,2}(Q))^T. \quad (\text{SI-3})$$

In this basis, $\Gamma_\Lambda^{(2)}$ is defined as

$$\left(\Gamma_\Lambda^{(2)} \right)_{ij}(Q, Q') = \frac{\delta^2 \Gamma_\Lambda}{\delta \chi_i(-Q) \delta \chi_j(Q')}, \quad (\text{SI-4})$$

which is a matrix in the discrete field index $i = 1, 2, 3, 4$ and in the continuous momentum variable $Q = (\omega, \mathbf{q})$ collecting frequency and spatial momentum. Accordingly, the trace in (SI-1) involves both an integration over momenta and a sum over internal indices.

$\Gamma_\Lambda^{(2)}(Q, Q')$ is conveniently decomposed into a constant part and a fluctuation part. The latter is a polynomial in momentum-dependent fields and, therefore, a non-diagonal matrix in momentum space. In contrast, the constant part is obtained by (i) inserting spatially constant field configurations, i. e., $\chi(Q) = \chi \delta(Q)$ in momentum space, and (ii) evaluating them at their equilibrium values in the ordered phase. These read

$$\chi(Q)|_{\text{eq}} = \left(\sqrt{2\rho_0}, 0, 0, 0 \right)^T \delta(Q). \quad (\text{SI-5})$$

(Without loss of generality we choose the condensate amplitude to be real.) As a result, the constant part is diagonal in momentum space,

$$P_\Lambda(Q) \delta(Q - Q') \equiv \Gamma_\Lambda^{(2)}(Q, Q')|_{\text{eq}}, \quad (\text{SI-6})$$

and is structured into retarded, advanced, and Keldysh blocks,

$$P_\Lambda(Q) = \begin{pmatrix} 0 & P^A(Q) \\ P^R(Q) & P^K(Q) \end{pmatrix}. \quad (\text{SI-7})$$

(For notational simplicity, we suppress the scale index Λ for the different blocks and their respective entries.) The retarded and advanced blocks are mutually hermitian conjugate (we decompose Z and \bar{K} into real and imaginary parts, $Z = Z_R + iZ_I$, $\bar{K} = \bar{A} + i\bar{D}$),

$$P^R(Q) = \begin{pmatrix} -iZ_I\omega - \bar{A}\mathbf{q}^2 - 2\text{Re}(\bar{u})\rho_0 & iZ_R\omega - \bar{D}\mathbf{q}^2 \\ -iZ_R\omega + \bar{D}\mathbf{q}^2 + 2\text{Im}(\bar{u})\rho_0 & -iZ_I\omega - \bar{A}\mathbf{q}^2 \end{pmatrix},$$

$$P^A(Q) = (P^R(Q))^\dagger. \quad (\text{SI-8})$$

Note that $\det P^R(Q = 0) = \det P^A(Q = 0) = 0$; the existence of a gapless mode associated to the broken $U(1)$ symmetry is thus ensured in our truncation at all scales Λ . For the Keldysh block we have

$$P^K = i\bar{\gamma}\mathbb{1}. \quad (\text{SI-9})$$

THE REGULATOR FUNCTION

The cutoff $\Delta\mathcal{S}_\Lambda$ is used in Eq. (3) of the main text to generate the effective action Γ_Λ from the microscopic action \mathcal{S} by suppressing contributions from momenta below Λ . Its second functional derivative $R_\Lambda = \Delta\mathcal{S}_\Lambda^{(2)}$ enters the exact flow equation (SI-1). We choose an optimized cutoff function [6] of the form

$$R_\Lambda(Q) = (\mathbf{q}^2 - \Lambda^2) \theta(\Lambda^2 - \mathbf{q}^2) \begin{pmatrix} 0 & R^R \\ R^A & 0 \end{pmatrix}, \quad (\text{SI-10})$$

where

$$R^R = \begin{pmatrix} -\bar{A} & -\bar{D} \\ \bar{D} & -\bar{A} \end{pmatrix}, \quad R^A = (R^R)^T. \quad (\text{SI-11})$$

Due to the θ -function in (SI-10), in the regularized inverse Green's function

$$G_\Lambda^{-1} = P_\Lambda + R_\Lambda, \quad (\text{SI-12})$$

momenta \mathbf{q}^2 smaller than the running scale Λ^2 acquire an effective mass $\propto \Lambda^2$ and we have $\det G_\Lambda^{-1}(Q=0) \neq 0$, which ensures that momentum integrals over Green's functions are infrared convergent. Note that it is sufficient for R_Λ to modify only the retarded and advanced blocks (i.e., the spectrum) of the inverse Green's function. The choice of a frequency-independent cutoff allows us to perform frequency integrals analytically.

The interpolation property of Γ_Λ between the classical action \mathcal{S} and the effective action Γ is guaranteed by the limiting behavior [7]

$$\lim_{\Lambda^2 \rightarrow \Lambda_0^2} R_\Lambda \sim \Lambda_0^2, \quad \lim_{\Lambda^2 \rightarrow 0} R_\Lambda = 0. \quad (\text{SI-13})$$

FLOW OF THE EFFECTIVE POTENTIAL

In equilibrium problems, an important object for practical calculations is the effective potential. It describes the homogeneous part of the effective action and is obtained by evaluating the full effective action at spatially homogeneous field configurations, $\bar{U} = \Gamma/\Omega|_{\chi(Q)=\chi\delta(Q)}$ (Ω is the quantization volume). In the framework of a derivative expansion, a closed flow equation can be derived for this object, which serves as a compact generating functional for the flow of all local couplings to arbitrarily high order. Here we provide the Keldysh analog of this construction, where the key difference roots in the occurrence of two field variables ϕ_c, ϕ_q , in contrast to a single field in equilibrium. However, for a theory which obeys the power counting discussed in the main text, we can parameterize the homogeneous part of the effective action as

$$\bar{V} = \frac{\partial \bar{U}}{\partial \phi_c} \phi_q + \frac{\partial \bar{U}^*}{\partial \phi_c^*} \phi_q^* + i\bar{\gamma} \phi_q^* \phi_q, \quad (\text{SI-14})$$

with $\bar{U} = \bar{U}(\phi_c^* \phi_c)$ dependent on the $U(1)$ invariant combination of classical fields only, this function thus being the direct counterpart of the effective potential. A flow equation can be derived for the auxiliary object \bar{V} , which reads (we introduce a dimensionless scale derivative $\partial_\ell \equiv \Lambda \partial_\Lambda$)

$$\partial_\ell \bar{V} = -\frac{i}{2} \int_Q \text{tr} [\mathcal{G}_\Lambda(Q) \partial_\ell R_\Lambda(Q)]. \quad (\text{SI-15})$$

Here, the inverse of \mathcal{G}_Λ is obtained from the full second functional variation by evaluating it at homogeneous field configurations (step (i) above Eq. (SI-5)), however without inserting the equilibrium values (step (ii)): $\mathcal{G}_\Lambda^{-1} = \Gamma_\Lambda^{(2)}|_{\chi(Q)=\chi\delta(Q)} + R_\Lambda$. \mathcal{G}_Λ is then diagonal in momentum space, and so the trace over momentum space in Eq. (SI-1) reduces to a single momentum integration, giving rise to the above compact form. In contrast to G_Λ^{-1} , \mathcal{G}_Λ^{-1} has a non-vanishing upper left block P^H . However, it vanishes when the background fields are set to their equilibrium values, $P^H|_{\text{eq}} = 0$, which is a manifestation of causality in the Keldysh formalism [4, 5].

From this equation we obtain the β -functions for the momentum-independent couplings by evaluating appropriate derivatives with respect to the $U(1)$ invariants

$$\rho_c = \phi_c^* \phi_c, \quad \rho_{cq} = \phi_c^* \phi_q = \rho_{qc}^*, \quad \rho_q = \phi_q^* \phi_q. \quad (\text{SI-16})$$

at their equilibrium values $\rho_c|_{\text{eq}} = \rho_0$, $\rho_{cq}|_{\text{eq}} = \rho_{qc}|_{\text{eq}} = \rho_q|_{\text{eq}} = 0$. Specifically, we use the projection prescriptions

$$\begin{aligned} \partial_\ell \rho_0 &= \beta_{\rho_0} = -\frac{1}{u} [\partial_{\rho_{cq}} \partial_\ell \bar{V}]_{\text{eq}}, \\ \partial_\ell \bar{u} &= \beta_{\bar{u}} = \bar{u}' \partial_\ell \rho_0 + [\partial_{\rho_c \rho_{cq}}^2 \partial_\ell \bar{V}]_{\text{eq}}, \\ \partial_\ell \bar{u}' &= \beta_{\bar{u}'} = [\partial_{\rho_c}^2 \partial_{\rho_{cq}} \partial_\ell \bar{V}]_{\text{eq}}, \\ \partial_\ell \bar{\gamma} &= \beta_{\bar{\gamma}} = i\rho_0 [\partial_{\rho_{cq} \rho_{qc}}^2 \partial_\ell \bar{V}]_{\text{eq}}. \end{aligned} \quad (\text{SI-17})$$

Calculation of the explicit expressions here and below is largely automatized using MATHEMATICA.

FLOW OF THE INVERSE PROPAGATOR

While the flow equation for the effective potential (SI-15) generates β -functions for all momentum-independent couplings, the flow of the complex dynamic Z and kinetic \bar{K} couplings, which constitute the momentum-dependent part of the effective action (5), is determined by the flow equation for the inverse propagator. We obtain the latter by taking the second variational derivative of the exact flow equation (SI-1) and setting the background fields to their equilibrium values

Eq. (SI-5),

$$\begin{aligned} \partial_\ell P_{\Lambda,ij}(Q) = & \frac{i}{2} \int_{Q'} \text{tr} [G_\Lambda^2(Q' - Q) \partial_\ell R_\Lambda(Q' - Q) \gamma_i G_\Lambda(Q) \gamma_j \\ & + G_\Lambda(Q' - Q) \gamma_i G_\Lambda^2(Q) \partial_\ell R_\Lambda(Q) \gamma_j], \end{aligned} \quad (\text{SI-18})$$

where

$$\gamma_{i,jl} \delta(P - P' + Q) = \frac{\delta \Gamma_{\Lambda,jl}^{(2)}(P, P')}{\delta \chi_i(Q)} \Big|_{\text{eq}}. \quad (\text{SI-19})$$

In Eq. (SI-18) we omit tadpole contributions $\propto \Gamma_\Lambda^{(4)}$, which do not depend on the external momentum Q and hence do not contribute to the flow of Z or \bar{K} . For these we use the projection prescriptions

$$\begin{aligned} \partial_\ell Z = \beta_Z = & -\frac{1}{2} \partial_\omega \text{tr} [(\mathbb{1} + \sigma_y) \partial_\ell P^R(Q)] \Big|_{Q=0}, \\ \partial_\ell \bar{K} = \beta_{\bar{K}} = & \partial_{\mathbf{q}^2} [\partial_\ell P_{22}^R(Q) + i \partial_\ell P_{12}^R(Q)] \Big|_{Q=0}. \end{aligned} \quad (\text{SI-20})$$

The β -functions (SI-17) and (SI-20) constitute the components of $\beta_{\mathbf{g}} = (\beta_Z, \beta_{\bar{K}}, \beta_{\rho_0}, \beta_{\bar{u}}, \beta_{\bar{u}'}, \beta_{\bar{\gamma}})^T$.

RESCALED FLOW EQUATIONS

We write the flow equation for the complex dynamic coupling Z in the form

$$\partial_\ell Z = -\eta_Z Z. \quad (\text{SI-21})$$

The anomalous dimension η_Z is an algebraic function of the rescaled couplings (6) and ρ_0 . The same applies to the β -functions of the latter,

$$\begin{aligned} \partial_\ell K = \beta_K = & \eta_Z K + \frac{1}{Z} \beta_{\bar{K}}, \\ \partial_\ell u = \beta_u = & \eta_Z u + \frac{1}{Z} \beta_{\bar{u}}, \\ \partial_\ell u' = \beta_{u'} = & \eta_Z u' + \frac{1}{Z} \beta_{\bar{u}'}, \\ \partial_\ell \gamma = \beta_\gamma = & (\eta_Z + \eta_Z^*) \gamma + \frac{1}{|Z|^2} \beta_{\bar{\gamma}}. \end{aligned} \quad (\text{SI-22})$$

In particular, the very right expressions in these equations ($\beta_{\bar{K}}/Z$ etc.) are functions of the rescaled couplings alone. In terms of these variables, therefore, all explicit reference to the running coupling Z is gone, and we have effectively traded the differential flow equation for Z for the algebraic expression for its anomalous dimension η_Z .

All couplings except for γ are complex valued. Taking real and imaginary parts of the β -functions for K , u , and u' yields the flow equations for A , D , λ , κ , λ' , and κ' respectively,

$$\begin{aligned} \partial_\ell A = \beta_A = \text{Re } \beta_K, \quad \partial_\ell D = \beta_D = \text{Im } \beta_K, \\ \partial_\ell \lambda = \beta_\lambda = \text{Re } \beta_u, \quad \partial_\ell \kappa = \beta_\kappa = \text{Im } \beta_u, \\ \partial_\ell \lambda' = \beta_{\lambda'} = \text{Re } \beta_{u'}, \quad \partial_\ell \kappa' = \beta_{\kappa'} = \text{Im } \beta_{u'}. \end{aligned} \quad (\text{SI-23})$$

The β -functions for the ratios $\mathbf{r} = (r_K, r_u, r_{u'})^T$ are then

$$\begin{aligned} \partial_\ell r_K = \beta_{r_K} = & \frac{1}{D} \beta_A - \frac{r_K}{D} \beta_D, \\ \partial_\ell r_u = \beta_{r_u} = & \frac{1}{\kappa} \beta_\lambda - \frac{r_u}{\kappa} \beta_\kappa, \\ \partial_\ell r_{u'} = \beta_{r_{u'}} = & \frac{1}{\kappa'} \beta_{\lambda'} - \frac{r_{u'}}{\kappa'} \beta_{\kappa'}. \end{aligned} \quad (\text{SI-24})$$

The number of flow equations can be further reduced by introducing anomalous dimensions for D and γ ,

$$\begin{aligned} \partial_\ell D = -\eta_D D, \\ \partial_\ell \gamma = -\eta_\gamma \gamma. \end{aligned} \quad (\text{SI-25})$$

As for the dynamic coupling Z in terms of the rescaled variables K , u , u' , γ and ρ_0 , all explicit reference to D and γ drops out, and we obtain for the couplings $\mathbf{s} = (w, \tilde{\kappa}, \tilde{\kappa}')^T$ defined in Eq. (7)

$$\begin{aligned} \partial_\ell w = \beta_w = & -(2 - \eta_D) w + \frac{w}{\kappa} \beta_\kappa + \frac{2\kappa}{\Lambda^2 D} \beta_{\rho_0}, \\ \partial_\ell \tilde{\kappa} = \beta_{\tilde{\kappa}} = & -(1 - 2\eta_D + \eta_\gamma) \tilde{\kappa} + \frac{\gamma}{2\Lambda D^2} \beta_\kappa, \\ \partial_\ell \tilde{\kappa}' = \beta_{\tilde{\kappa}'} = & -(-3\eta_D + 2\eta_\gamma) \tilde{\kappa}' + \frac{\gamma^2}{4D^3} \beta_{\kappa'}. \end{aligned} \quad (\text{SI-26})$$

In summary, the transformations (6) and (7) result in the closed system (8) for \mathbf{r} and \mathbf{s} with $\beta_{\mathbf{r}} = (\beta_{r_K}, \beta_{r_u}, \beta_{r_{u'}})^T$ given by Eq. (SI-24) and $\beta_{\mathbf{s}} = (\beta_w, \beta_{\tilde{\kappa}}, \beta_{\tilde{\kappa}'})^T$ given by Eq. (SI-26). The flows of Z , D , and γ are decoupled and determined by the anomalous dimensions (SI-21) and (SI-25), which are themselves functions of \mathbf{r} and \mathbf{s} .

CRITICAL PROPERTIES

For the analysis of critical behavior, we need to find a scaling solution to the flow equations for the bare couplings or, equivalently, a fixed point \mathbf{r}_* , \mathbf{s}_* of the flow of dimensionless rescaled couplings,

$$\beta_{\mathbf{r}}(\mathbf{r}_*, \mathbf{s}_*) = \beta_{\mathbf{s}}(\mathbf{r}_*, \mathbf{s}_*) = \mathbf{0}. \quad (\text{SI-27})$$

This non-linear algebraic set of equations has a non-trivial solution given by Eq. (9). In order to characterize the infrared flow in the vicinity of the fixed point (encoding the critical exponents we are interested in here), we study the flow of the couplings linearized around the fixed point, cf. Eq. (10). The stability matrices N and S in this equation read explicitly

$$N = \nabla_{\mathbf{r}}^T \beta_{\mathbf{r}} \Big|_{\mathbf{r}=\mathbf{r}_*, \mathbf{s}=\mathbf{s}_*} = \begin{pmatrix} 0.0525 & 0.0586 & 0.0317 \\ -0.0002 & -0.0526 & 0.1956 \\ 0.4976 & -2.3273 & 1.9725 \end{pmatrix}, \quad (\text{SI-28})$$

$$S = \nabla_{\mathbf{s}}^T \beta_{\mathbf{s}} \Big|_{\mathbf{r}=\mathbf{r}_*, \mathbf{s}=\mathbf{s}_*} = \begin{pmatrix} -1.6204 & 0.0881 & 0.0046 \\ -3.1828 & 0.2899 & 0.0363 \\ -15.3743 & -42.2487 & 2.1828 \end{pmatrix}, \quad (\text{SI-29})$$

without coupling between \mathbf{r} and \mathbf{s} sectors. At present we cannot rule out that an extended truncation would couple them. However, since we already include all relevant and marginal couplings, we expect the decoupling to be robust or at least approximately valid to a good accuracy.

The infrared flow of Z , D , and γ is determined by the values of the respective anomalous dimensions at the fixed point. Equations (SI-21) and (SI-25) imply the scaling behavior

$$Z \sim \Lambda^{-\eta_Z}, \quad D \sim \Lambda^{-\eta_D}, \quad \gamma \sim \Lambda^{-\eta_\gamma} \quad (\text{SI-30})$$

for $\Lambda \rightarrow 0$. While η_D and η_γ describe the flow of real quantities and are, therefore, themselves real by definition, η_Z is in general a complex valued function of \mathbf{r} and \mathbf{s} . At the fixed point, however, the imaginary part vanishes,

$$\text{Im } \eta_Z = 0, \quad (\text{SI-31})$$

which ensures scale invariance of the full effective action at the critical point.

As is indicated in the main text, the emergence of $O(2)$ model critical properties in the sector \mathbf{s} is due to the scaling relation $\eta_Z = -\eta_\gamma$, which ensures that these anomalous dimensions compensate each other in the β -functions for the couplings \mathbf{s} . This can be seen most simply by ex-

pressing, e. g., $\tilde{\kappa}$ in terms of bare quantities,

$$\tilde{\kappa} = \frac{\gamma \text{Im}(u)}{2\Lambda \text{Im}(K)^2} = \frac{\gamma \text{Im}(\bar{u}/Z)}{2\Lambda \text{Im}(\bar{K}/Z)^2}. \quad (\text{SI-32})$$

In this form it is apparent that the scaling $\sim \Lambda^{-\eta_Z}$ which applies to both Z and $1/\gamma$ drops out. Similar arguments hold for w and $\tilde{\kappa}'$. Alternatively, the cancellation of η_Z and η_γ in the β -functions can be seen explicitly by inserting Eqs. (SI-22) and (SI-23) in (SI-26). What remains is a dependence on $\eta \equiv \eta_D + \eta_Z$ which is just the anomalous dimension associated with the bare kinetic coefficient \bar{K} .

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